J. Fluid Mech. (1982), *vol.* 116, *pp.* 115–121
Printed in Great Britain

Information decay and the predictability of turbulent flows

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(Received 13 April 1981 and in revised form 14 July 1981)

A measure of predictability that has many superior features compared to currently used measures is introduced. Through statistical theory it is demonstrated that in inviscid truncated flow this new predictability measure increases monotonically in time while all initial information about the system decays. Under the influence of forcing and viscosity the behaviour of this measure is shown always to satisfy intuitive expectations.

1. Introduction

In this paper we develop the very natural synthesis between two recent developments in statistical fluid mechanics. Firstly we recall theoretical work on the predictability of fluid motions, i.e. the question of how well future states of a fluid may be forecast given some perfect dynamical model but with an imperfect specification of initial conditions. Secondly, we consider the role of entropy or 'information' in describing macroscale fluid motions. Then we show that a suitable entropy may be defined which measures the 'uncertainty' regarding fluid states and which satisfies the property that entropy, defined for an ensemble of realizations, does not decrease in time except through external couplings. We discuss the roles of external couplings – both the dissipative coupling to the fields of microscale motion or radiation, and coupling to imposed forces. Such discussion is not limited to fully developed turbulence but rather may include wave propagation, as in the case of the role of planetary waves in weather predictability.

2. Statistical theory

Predictability of fluid motion has been considered by Thompson (1957), Charney *et al.* (1966), Lorenz (1963, 1969), Smagorinsky (1969), Kraichnan (1970), Leith (1971), Lilly (1972) and Leith & Kraichnan (1972), among others. We imagine two realizations of fluid motion which differ initially by a small, random 'error' field. Under some circumstances, that error grows in time until the two realizations become completely uncorrelated. This concept of predictability generalizes the idea of flow instability to circumstances where the 'unperturbed' flow may be a complicated function of space and time. Predictability theory seeks to describe the evolution of the variance error field averaged over an ensemble of realizations of pairs of flows.

Leith & Kraichnan (1972) present a theory of predictability for isotropic, homogeneous, incompressible, turbulent flow based on Markovian closure theory (cf. Orszag 1970). This theory provides evolution equations for the variance spectra of the Fourier amplitudes of the two velocity fields, $u_i^{(1)}(\mathbf{k},t)$ and $u_i^{(2)}(\mathbf{k},t)$, in the pair of flows to be compared. The ensemble for field $u_i^{(1)}(\mathbf{k},t)$ is assumed statistically identical to that for field $u_i^{(2)}(\mathbf{k},t)$, and the average field is assumed to vanish. The variance spectra for these fields are defined by (assuming reflection symmetry)

$$P_{ij}(\mathbf{k})\,U_{\mathbf{k}} \equiv V^{-1}\langle u_i^{(m)}(\mathbf{k})\,u_j^{(m)}(-\mathbf{k})\rangle, \tag{2.1a}$$

$$P_{ij}(\mathbf{k})\,W_{\mathbf{k}} \equiv V^{-1}\langle u_i^{(m)}(\mathbf{k})\,u_j^{(n)}(-\mathbf{k})\rangle \quad (m \neq n), \tag{2.1b}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij}-k_i k_j/k^2$, and V is the k-space volume element $(2\pi/L)^d$ for the discrete Fourier transform in a d-dimensional box of side L. Markovian two-point closure theory (cf. Orszag 1970; Fournier & Frisch 1978; Rose & Sulem 1978; Carnevale, Frisch & Salmon 1981) predicts the rate of change of these spectra as

$$\dot{U}_{\mathbf{k}} = \frac{2V^2}{d-1} \sum_{\substack{\mathbf{p},\mathbf{q} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=0)}} \theta_{\mathbf{kpq}} \frac{\sin^2\alpha}{k^2}[(p^2-q^2)(k^2-q^2)+(d-2)\,k^2p^2] \times [U_{\mathbf{p}}U_{\mathbf{q}}-U_{\mathbf{k}}U_{\mathbf{q}}]-2\nu_{\mathbf{k}}U_{\mathbf{k}}+F_{\mathbf{k}}, \tag{2.2a}$$

$$\dot{W}_{\mathbf{k}} = \frac{2V^2}{d-1} \sum_{\substack{\mathbf{p},\mathbf{q} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=0)}} \theta_{\mathbf{kpq}} \frac{\sin^2\alpha}{k^2}[(p^2-q^2)(k^2-q^2)+(d-2)\,k^2p^2] \times [W_{\mathbf{p}}W_{\mathbf{q}}-W_{\mathbf{k}}U_{\mathbf{q}}]-2\nu_{\mathbf{k}}W_{\mathbf{k}}+R_{\mathbf{k}}. \tag{2.2b}$$

$F_{\mathbf{k}}$ and $R_{\mathbf{k}}$ represent the effects of external, stochastic, Gaussianly distributed forcing $f_i^{(m)}(t)$ which is white noise in time; they are defined by

$$P_{ij}(\mathbf{k})\,F_{\mathbf{k}} = V^{-1}\langle f_i^{(m)}(\mathbf{k})f_j^{(m)}(-\mathbf{k})\rangle, \tag{2.3a}$$

$$P_{ij}(\mathbf{k})\,R_{\mathbf{k}} = V^{-1}\langle f_i^{(m)}(\mathbf{k})f_j^{(n)}(-\mathbf{k})\rangle \quad (m \neq n). \tag{2.3b}$$

$\nu_{\mathbf{k}}$ represents a generalized 'dissipation' function. Dissipative processes such as viscosity can be represented by $\nu_{\mathbf{k}} > 0$, while instability-type forcing processes can be represented with $\nu_{\mathbf{k}} < 0$. α represents the angle between wave vectors $\mathbf{p}$ and $\mathbf{q}$. θ_{kpq} is the triad relaxation time for the interaction of the modes $\mathbf{k}$, $\mathbf{p}$ and $\mathbf{q}$; it is positive, symmetric under permutation of $\mathbf{k}$, $\mathbf{p}$ and $\mathbf{q}$ (cf. Rose & Sulem 1978), and for present purposes otherwise arbitrary.

The equations (2.2) are valid for arbitrary dimension $d \geqslant 2$ (Fournier & Frisch 1978; Rose & Sulem 1978). Neglecting external forces they are the equations studied by Leith & Kraichnan (1972) for $d = 2$ or 3. Leith & Kraichnan (1972) also take the limit $L \to \infty$, but for purposes of possible comparison with computer simulation we remain in the discrete notation and for convenience assume unit normalization, $V \equiv 1$.

The error spectrum is defined by $\Delta_{\mathbf{k}} = U_{\mathbf{k}}-W_{\mathbf{k}}$. Lack of correlation, $W_{\mathbf{k}} = 0$, of the comparison fields corresponds to $\Delta_{\mathbf{k}} = U_{\mathbf{k}}$, what we refer to as 'complete ignorance'. Perfect correlation, $W_{\mathbf{k}} = U_{\mathbf{k}}$, corresponds to $\Delta_{\mathbf{k}} = 0$, what we refer to as 'complete knowledge'. Perfect anticorrelation, $W_{\mathbf{k}} = -U_{\mathbf{k}}$, would also correspond to complete knowledge, but is not a particularly interesting situation. The Markovian closure equation for $\Delta_{\mathbf{k}}$ is obtained from (2.2) and is

$$\dot{\Delta}_{\mathbf{k}} = \frac{2}{d-1} \sum_{\substack{\mathbf{p},\mathbf{q} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=0)}} \theta_{\mathbf{kpq}} \frac{\sin^2\alpha}{k^2} [(p^2-q^2)(k^2-q^2)+(d-2)k^2p^2] \times [\Delta_{\mathbf{p}} U_{\mathbf{q}} - U_{\mathbf{q}}\Delta_{\mathbf{k}} + W_{\mathbf{p}}\Delta_{\mathbf{q}}] - 2\nu_{\mathbf{k}}\Delta_{\mathbf{k}} + F_{\mathbf{k}} - R_{\mathbf{k}}. \quad (2.4)$$

A couple of simple observations from (2.4) are immediate. If there is no initial uncertainty and $R_{\mathbf{k}} = F_{\mathbf{k}}$ for all $\mathbf{k}$, then $W_{\mathbf{k}} = U_{\mathbf{k}}$, $\Delta_{\mathbf{k}} = 0$, and such complete knowledge persists. If initial uncertainty is total, then $\Delta_{\mathbf{k}} = U_{\mathbf{k}}$, $W_{\mathbf{k}} = 0$, and complete ignorance persists. When $\nu_{\mathbf{k}} > 0$, dissipation provides for decay of $\Delta_{\mathbf{k}}$ along with decay of $U_{\mathbf{k}}$ or $W_{\mathbf{k}}$. If external forces are uncorrelated between realizations then $R_{\mathbf{k}} = 0$, and $F_{\mathbf{k}}$ is a source for $\Delta_{\mathbf{k}}$. If the same external force is applied to each realization then $R_{\mathbf{k}} = F_{\mathbf{k}}$, and $\Delta_{\mathbf{k}}$ is unaffected.

Beyond these simple observations very little can apparently be said. If we consider the spectrally truncated system (i.e. only a finite number of modes) and set $\nu_{\mathbf{k}} = F_{\mathbf{k}} = 0$ then we are dealing with closed conservative systems. In that case, we might expect from a general statistical-mechanical or information-theoretical principle that 'uncertainty' cannot decrease. A question is how to interpret this principle. From (2.4) we can easily find circumstances where $\Delta_{\mathbf{k}}$ decreases at some $\mathbf{k}$. Neither is there any apparent wavenumber-weighted sum over $\Delta_{\mathbf{k}}$ which does not decrease. This paradox, to reconcile predictability theory as given by (2.4) with the second law of thermodynamics, motivates the second part of our paper.

3. Entropy and the predictability H-theorem

The close relation between statistical theories of turbulence as developed by Kraichnan (1959), Edwards (1964), Herring (1965) and others, and methods of disequilibrium statistical mechanics as described e.g. by Prigogine (1962) has encouraged various authors, including Edwards & McComb (1969), Cook (1974), Montgomery (1976) and Carnevale *et al.* (1981) to consider the role of macroscale entropy which, for d-dimensional homogeneous isotropic turbulence, has the expression

$$S = \tfrac{1}{2}(d-1)\sum_{\mathbf{k}} \ln U_{\mathbf{k}}. \quad (3.1)$$

In more general cases where flow may be correlated with external fields, such as quasi-geostrophic turbulence above irregular topography, or where internal correlations as among vertical layers in a layer model of quasi-geostrophic turbulence are to be considered, Carnevale *et al.* (1981) have demonstrated an important extension of (3.1). Consider an overall second moment correlation matrix $\mathbf{Y}$ among all of the wave-vector coefficients of all of the fields present. The appropriate entropy then is

$$S = \tfrac{1}{2}\ln\det\mathbf{Y}. \quad (3.2)$$

Carnevale *et al.* (1981) show that S defined by (3.2) is related to our lack of information about the ensemble, and that, assuming Gaussian initial conditions, S cannot decrease in time for conservative systems. Furthermore, they demonstrate very generally that second-order Markovian closure implies that (3.2) satisfies a Boltzmann-like H-

theorem. That is $dS/dt = 0$ for the canonical equilibrium ensemble, but otherwise $dS/dt > 0$. This result is non-trivial, especially since the underlying Eulerian dynamical equations are non-Hamiltonian. Corresponding H-theorems for case of weakly inter-acting waves as discussed by Hasselmann (1966) follow more directly from the Hamiltonian dynamical basis for such systems.

The role of entropy in the description of macroscale fluid motions has tended to be viewed as esoteric. Entropy as given by (3.2) does not have the same intuitive reality as kinetic energy, say. Yet it is intriguing that calculation in ocean basins, on a rotating sphere, in two-layer fluids and above arbitrary topography, which are based upon maximizing S subject only to overall constraints such as total energy, total enstrophy or angular momentum, produce some strikingly realistic flows, as in Salmon, Holloway & Hendershott (1976) or Frederiksen & Sawford (1980).

When we turn to predictability, the role of entropy becomes natural and indeed, we feel, essential. In the case of weather predictability, we are not so concerned about the future kinetic energy of the atmosphere nor even the kinetic energy of the 'error wind'. Rather, we seek a quantitative measure of the maximum information that we may hope to provide about future states of the atmosphere, given our current information about the atmosphere. At once (3.2) will provide such a measure and also resolve the second-law paradox to which we alluded above.

When we apply prescription (3.2) to the problem at hand we find

$$S = \tfrac{1}{2}(d-1)\sum_{\mathbf{k}} \ln (U_{\mathbf{k}}^2 - W_{\mathbf{k}}^2), \tag{3.3a}$$

up to uninteresting additive constants (for details see Carnevale *et al.* 1981; Carnevale 1979). Or we could rewrite (3.3*a*) in terms of $\Delta_{\mathbf{k}}$ and a new variable $\sigma_{\mathbf{k}} = U_{\mathbf{k}} + W_{\mathbf{k}}$ as

$$S = \tfrac{1}{2}(d-1)\sum_{\mathbf{k}} (\ln \Delta_{\mathbf{k}} + \ln \sigma_{\mathbf{k}}). \tag{3.3b}$$

Since in the state of complete ignorance ($W_{\mathbf{k}} = 0$) the error field $\Delta_{\mathbf{k}}$ is not a maximum but rather takes the intermediate value $U_{\mathbf{k}}$, it is quite natural that S depends rather on $U_{\mathbf{k}}^2 - W_{\mathbf{k}}^2$, which takes on its maximum value $U_{\mathbf{k}}^2$ in the state of complete ignorance. The greater our ignorance, the larger is $U_{\mathbf{k}}^2 - W_{\mathbf{k}}^2$ and hence the larger is S.

Since the Leith & Kraichnan (1972) equations (2.2) with $\nu_{\mathbf{k}} = F_{\mathbf{k}} = 0$ are second-order Markovian equations they must, according to Carnevale *et al.* (1981), imply that S increases monotonically with $U_{\mathbf{k}}$ and $W_{\mathbf{k}}$ approaching canonical equilibrium. The canonical-equilibrium spectra are

$$U_{\mathbf{k}}^{\mathrm{eq}} = (a + bk^2)^{-1}, \quad W_{\mathbf{k}}^{\mathrm{eq}} = 0, \tag{3.4a, b}$$

where $b = 0$ for $d > 2$ (cf. Salmon *et al.* 1976; Carnevale *et al.* 1981). S increases monotonically toward its equilibrium value, which corresponds to the state of complete ignorance $\Delta_{\mathbf{k}} = U_{\mathbf{k}}$.

To include the effects of external forcing and generalized dissipation, we derive the evolution equation for S directly from (2.2). To write the equation for

$$\dot{S} = (d-1)\sum_{\mathbf{k}} \frac{U_{\mathbf{k}}\dot{U}_{\mathbf{k}} - W_{\mathbf{k}}\dot{W}_{\mathbf{k}}}{U_{\mathbf{k}}^2 - W_{\mathbf{k}}^2} = \tfrac{1}{2}(d-1)\sum_{\mathbf{k}} \frac{\dot{\Delta}_{\mathbf{k}}}{\Delta_{\mathbf{k}}} + \frac{\dot{\sigma}_{\mathbf{k}}}{\sigma_{\mathbf{k}}} \tag{3.5}$$

which derives from (2.2), we first note that $k^{-2}\sin^2\alpha$ is symmetric under permutation of **k**, **p** and **q**, and we recall that $\theta_{\mathbf{kpq}}$ is also symmetric. Then by changes in dummy-summation variables and some tedious algebraic manipulation we obtain

$$\begin{aligned}\frac{dS}{dt} = \frac{1}{4}\sum_{\substack{\mathbf{k},\mathbf{p},\mathbf{q}\\ \mathbf{k}+\mathbf{p}+\mathbf{q}=0}} \theta_{\mathbf{kpq}}\frac{\sin^2\alpha}{k^2}\Bigg\{&\tfrac{1}{3}\Delta_{\mathbf{k}}\Delta_{\mathbf{p}}\Delta_{\mathbf{q}}\left(\frac{p^2-q^2}{\Delta_{\mathbf{k}}}+\frac{q^2-k^2}{\Delta_{\mathbf{p}}}+\frac{k^2-p^2}{\Delta_{\mathbf{q}}}\right)^2\\ &+\Delta_{\mathbf{k}}\sigma_{\mathbf{p}}\sigma_{\mathbf{q}}\left(\frac{p^2-q^2}{\Delta_{\mathbf{k}}}+\frac{q^2-k^2}{\sigma_{\mathbf{p}}}+\frac{k^2-p^2}{\sigma_{\mathbf{q}}}\right)^2\\ &+(d-2)\,k^2p^2\left[\Delta_{\mathbf{k}}\Delta_{\mathbf{p}}\Delta_{\mathbf{q}}\left(\frac{1}{\Delta_{\mathbf{k}}}-\frac{1}{\Delta_{\mathbf{p}}}\right)^2+2\Delta_{\mathbf{k}}\sigma_{\mathbf{p}}\sigma_{\mathbf{q}}\left(\frac{1}{\Delta_{\mathbf{k}}}-\frac{1}{\sigma_{\mathbf{p}}}\right)^2\right.\\ &\left.+\sigma_{\mathbf{k}}\sigma_{\mathbf{p}}\Delta_{\mathbf{q}}\left(\frac{1}{\sigma_{\mathbf{k}}}-\frac{1}{\sigma_{\mathbf{p}}}\right)^2\right]\Bigg\}-2(d-1)\sum_{\mathbf{k}}\nu_{\mathbf{k}}\\ &+(d-1)\sum_{\mathbf{k}}\frac{U_{\mathbf{k}}F_{\mathbf{k}}-W_{\mathbf{k}}R_{\mathbf{k}}}{U_{\mathbf{k}}^2-W_{\mathbf{k}}^2}. \end{aligned} \tag{3.6}$$

By the Schwartz inequality $0 \leqslant \Delta_{\mathbf{k}}$ and $0 \leqslant \sigma_{\mathbf{k}}$. Thus for $\nu_{\mathbf{k}} = 0$, dS/dt is manifestly non-negative. Furthermore, assuming $U_{\mathbf{k}}^2 - W_{\mathbf{k}}^2 > 0$ initially, so that S is always well-defined, it can be shown in the case of $\nu_{\mathbf{k}} = F_{\mathbf{k}} = 0$ that the only analytic solution to $dS/dt = 0$ is the canonical equilibrium result (3.4).

By the Schwartz inequality the external forcing term in (3.6) is always non-negative. It vanishes only when the systems and the forcing realizations are perfectly correlated, but is otherwise positive. This is what we would intuitively expect to be the effect of random forcing on uncertainty.

If $\nu_{\mathbf{k}} > 0$ for all **k** then the dissipative term tends to decrease S. This is also as one would intuitively expect, because viscosity drives the systems toward the perfectly predictable state of zero motion. A negative value of $\nu_{\mathbf{k}}$ on the other hand contributes to the increase of S. This too is intuitive since a negative $\nu_{\mathbf{k}}$ acts as a random force with power $2|\nu_{\mathbf{k}}|\,U_{\mathbf{k}}$.

An interesting case is that in which the last two terms in (3.6) cancel each other exactly at all times. Then $dS/dt \geqslant 0$, and the system is again driven to a state with spectra (3.4). In particular, this occurs in the exact stationary, forced, viscous state examined by Thompson (1972) for which $2\nu_{\mathbf{k}} = F_{\mathbf{k}}/U_{\mathbf{k}}^{\mathrm{eq}}$, $W_{\mathbf{k}} = 0$ and $R_{\mathbf{k}} = 0$.

A question of great concern in atmospheric predictability is the effect of a differential rotation rate. There are theoretical arguments and numerical simulation evidence to suggest that predictability is enhanced for flow on a β-plane (Basdevant *et al.* 1981; Carnevale 1981; Holloway 1981). The Markovian closure equations (2.2) for $d = 2$ are valid for anisotropic as well as isotropic flow (simply replace wavenumber variables with wave vectors). The effect of differential rotation enters these equations only through modifications of the triad relaxation time $\theta_{\mathbf{kpq}}$ – in general, the larger β is the smaller $\theta_{\mathbf{kpq}}$ is for a given spectrum (cf. Holloway & Hendershott 1977). Thus the tendency for S to increase as given by (3.6) is correspondingly reduced, suggesting enhanced predictability on the β-plane.

4. Discussion

We have introduced a measure of flow-predictability that satisfies certain compelling principles and intuitions. S defined by (3.3) for inviscid unforced systems must increase monotonically, representing a monotonic tendency to complete ignorance in

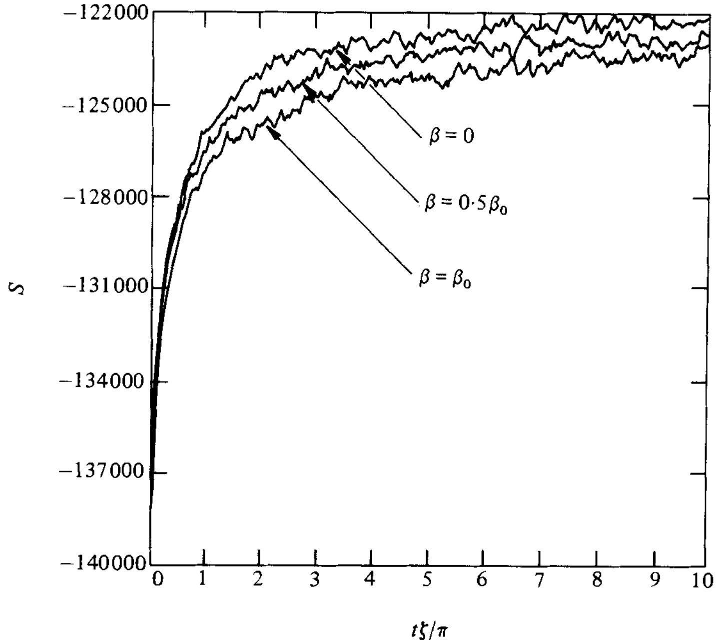

FIGURE 1. The evolution of S (3.3) for inviscid unforced simulation with differential rotation $\beta = 0$, $\beta = 0{\cdot}5\beta_0$ and $\beta = \beta_0$. The units of time are given in terms of a 'turnover time' π/ζ, with ζ the r.m.s. vorticity.

accord with the second law of thermodynamics. The tendencies of viscosity to decrease S and random forcing to increase S are in accord with intuition. The effect of differential rotation in slowing down the decay of predictability is in agreement with current studies on the subject. These properties of S cannot be matched by any spectrally weighted sum over the error field $\Delta_{\mathbf{k}}$.

As a practical matter we believe S can be a useful measure of predictability in comparing a flow with prediction. Although in comparing such a pair we only have a single realization estimate of $U_{\mathbf{k}}$ and $W_{\mathbf{k}}$, there is evidence to suggest S will still behave much like the ensemble predictions. For example, Carnevale (1981) has demonstrated that in simulation of two-dimensional flow initially far from equilibrium the behaviour of the entropy (3.1) agrees very closely with the predictions of closure and statistical mechanics.

We have done some preliminary work in which we simulate a pair of unforced, inviscid realizations and compute S as the spectra tend to equilibrium. The behaviour of the entropy for three experiments ($\beta = 0$, $\beta = 0{\cdot}5\beta_0$, and $\beta = \beta_0$) is shown in figure 1 for a simulation with resolution $(64)^2$. Here the effect of β relative to advective effects is measured by $\beta_0 = \zeta^2/u$, where ζ is the r.m.s. vorticity and u is the r.m.s. velocity (Rhines 1975). For $\beta \ll \beta_0$, effects of β are slight. For $\beta \gg \beta_0$, effects of β are dominant. We show the moderate cases $\beta = 0{\cdot}5\beta_0$ and $\beta = \beta_0$. The delay in the increase of uncertainty for $\beta = 0{\cdot}5\beta_0$ and β_0 compared to $\beta = 0$ is qualitatively as suggested in §3.

Of course, for single realizations noise appears in the value of S. However, both the near-equilibrium mean value of S and the size of the noise band can be accurately predicted from statistical mechanics, as shown by Carnevale (1981).

We are grateful to Philip D. Thompson for very helpful discussions. G. H. has been funded in part by the National Science Foundation, Office of GARP, under grant ATM 791064. This is University of Washington contribution number 1224.

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